Symmetry reduced Einstein gravity and generalized σ and chiral models

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Abstract

Certain features associated with the symmetry reduction of the vacuum Einstein equations by two commuting, space-like Killing vector fields are studied. In particular, the discussion encompasses the equations for the Gowdy T^3 cosmology and cylindrical gravitational waves. We first point out a relation between the $SL(2,R)$ (or $SO(3)$) σ and principal chiral models, and then show that the reduced Einstein equations can be obtained from a dimensional reduction of the standard $SL(2,R)$ σ–model in *three* dimensions. The reduced equations can also be derived from the action of a 'generalized' *two* dimensional $SL(2,R)$ σ–model with a time dependent constraint. We give a Hamiltonian formulation of this action, and show that the Hamiltonian evolution equations for certain phase space variables are those of a certain generalization of the principal chiral model. Using these Hamiltonian equations, we give a prescription for obtaining an infinite set of constants of motion explicitly as functionals of the space-time metric variables.

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I. INTRODUCTION

A central mathematical problem in non-linear field theories of physical interest is that of gaining 'control' over the space of solutions. One strategy is to map the theory of interest in to 'standard' models which have been well-analyzed. Another is to find the 'full set' of conserved quantities associated with field equations. Both these strategies have proved to be useful in two-dimensional field theories, where a number of integrable models have been known to exist for some time [1,2]. Now, in presence of two commuting, space-like Killing vector fields, the four dimensional, vacuum Einstein theory reduces to a two dimensional field theory. Hence, it is natural to explore the structure of this sector of general relativity using techniques that have been successful in standard integrable models.

If the Killing fields under consideration are hypersurface orthogonal, the theory has only one local degree of freedom, and the field equations can in effect be reduced to a rather simple linear partial differential equation. The model is then exactly soluble and one can explicitly write down the full set of conserved quantities [3–5]. On the other hand, when the Killing fields fail to be hypersurface orthogonal, the theory is both more difficult and more interesting: Now, there are *two* local degrees of freedom which remain coupled non-linearly. The space-times in this category include the Gowdy cosmological models [6] and cylindrical gravitational waves (see [7] and references therein). In this paper, we shall focus on these models.

Various aspects of the underlying equations have been studied in detail over the years (see, e.g., [8–15]). The results have led to a wealth of insights in to the structure of the space of solutions of the reduced theory which, in turn, have made these 'midi-superspaces' convenient tools to probe physical issues that arise in the full theory, such as quantization and cosmic censorship. Quantization of the one polarization cylindrical waves was studied in [7] and more recently, in [16]. The second analysis was subsequently used to show that, in this model, there exist surprisingly large quantum gravity effects which undermine the validity of the classical and semi-classical theories in unexpected ways [17]. More recently, a quantization of the more interesting two-polarization gravitational waves has also been achieved [18] and its physical implications are now being explored. Finally, the models have proved to be useful in analyzing cosmic censorship in cosmological [19] as well as asymptotically flat contexts [20].

The investigations of [8–15] have also shed light on the mathematical structure of the reduced Einstein equations. In particular, the presence of the infinite dimensional Geroch group which acts transitively on the space of solutions indicates that the system is in some sense exactly integrable. The application of the inverse scattering methods has also led to results supporting this view point. More recently there has been work on obtaining constants of motion for the reduced Einstein equations with open spatial topology in [21] and closed spatial topology in [22]. However, it is fair to say that we do not yet know if there exists a *complete* set of constants of motion; unlike in the standard, two-dimensional integrable models [1,2], an analysis involving, e.g., bi-symplectic structures has not been carried out. Hence it is of interest to seek new avenues to this problem.

The purpose of this paper is two fold: i) to point out new relations between the reduced Einstein system and the other well-known models which might be useful in future investigations; and, ii) to propose a new strategy to address the problem of finding explicit

constants of motion. More precisely, we will study the reduced Einstein equations in the metric variables, further develop the interplay between these equations and the standard and generalized σ–models in three and two dimensions respectively, and, using this relation, provide a new avenue for obtaining an infinite set of constants of motion.

The plan of the paper is as follows. In Sec. II we recall the reduced Einstein equations for the Gowdy T^3 cosmology and cylindrical gravitational waves. In Sec. III, we first describe a relation between the $SL(2,R)$ or $SO(3)$ σ–models and the principal chiral model in any space-time dimension, and then show that the reduced Einstein system is a subsystem of the standard, *three* dimensional $SL(2,R)$ σ–model. In Sec. IV we elaborate on the relation between a generalized σ–model in *two* dimensions, and present a Hamiltonian formulation and associated evolution equations. This framework is utilized in Sec. V to give a prescription for obtaining an infinite set of constants of motion.

II. PRELIMINARIES: REDUCED EINSTEIN EQUATIONS

Recall that solutions to the vacuum Einstein equations with two commuting, space-like, Killing vectors fall into classes of well known space-times. These include plane, cylindrical [7] and 'toroidal' [23] gravitational waves for non–compact spatial topology and the T^3, $S^1 \times S^2$ and S^3 Gowdy cosmological models for compact spatial topology [6]. For definiteness, we will consider one representative from each class, namely the T^3 cosmological model and cylindrical gravitational waves. Although this material is well-known, it is included here because the final results will serve as points of departure for our main analysis.

A. Two polarization Gowdy model

The T^3 Gowdy model has the space-time metric

$$ds^2 = e^{2A} \left(-dt^2 + d\theta^2 \right) + g_{ab} \, dx^a dx^b \;, \tag{2.1}$$

where $A = A(t,\theta)$, $x^a = (x^1, x^2)$, and the 2×2 metric

$$g_{ab} = R \begin{pmatrix} \cosh W + \cos\Phi \, \sinh W & \sin\Phi \, \sinh W \\ \sin\Phi \, \sinh W & \cosh W - \cos\Phi \, \sinh W \end{pmatrix} \tag{2.2}$$

is parametrized by the three functions $R(t,\theta)$, $\Phi(t,\theta)$ and $W(t,\theta)$. The metric (2.1) has two commuting space-like Killing vector fields

$$\left(\frac{\partial}{\partial x^1} \right) \quad \text{and} \quad \left(\frac{\partial}{\partial x^2} \right) \;, \tag{2.3}$$

with the additional restriction that each of their orbits has the topology of a two torus. The t, θ coordinates have ranges $0 < t < \infty$ and $0 \le \theta \le 2\pi$. With these conventions, the metric is that of the 3-torus Gowdy cosmology [6]. The vacuum Einstein equations in the Gowdy gauge, defined by setting $R(t,\theta) = t$, give the two coupled two-dimensional evolution equations

$$\ddot{W} + \frac{1}{t}\dot{W} - W'' + \sinh W \, \cosh W \, (\Phi'^2 - \dot{\Phi}^2) = 0 \, , \qquad (2.4)$$

$$\ddot{\Phi} + \frac{1}{t}\dot{\Phi} - \Phi'' + 2\,\frac{\cosh W}{\sinh W}\,(\dot{\Phi}\dot{W} - \Phi'W') = 0 \, , \qquad (2.5)$$

for $W(t,\theta)$ and $\Phi(t,\theta)$, and the two 'constraint' equations

$$\dot{A} + \frac{1}{4t} - \frac{t}{4}\,[\,\dot{W}^2 + W'^2 + \sinh^2 W\,(\dot{\Phi}^2 + \Phi'^2)\,] = 0 \, , \qquad (2.6)$$

$$A' - \frac{t}{2}\,(\dot{W}W' + \sinh^2 W\,\dot{\Phi}\Phi') = 0 \, . \qquad (2.7)$$

Note that the evolution equations involve only $W(t,\theta)$ and $\Phi(t,\theta)$ and, given a solution to these equations, the field $A(t,\theta)$ can be obtained by a simple integration of the constraint equations. (Eqns. (2.4), (2.5) also serve as the consistency conditions for the two constraints.)

However, because the spatial topology is compact, there is a subtlety: We can solve for A consistently only if the integral over the circle of the expression defining A' in Eqn. (2.7) vanishes. [1] This leads to a global constraint relating the two fields $W(t,\theta)$ and $\Phi(t,\theta)$, namely,

$$\int_0^{2\pi} d\theta\,(\dot{W}W' + \sinh^2 W\,\dot{\Phi}\Phi') = 0 \, , \qquad (2.8)$$

which is preserved by the evolution equations. Thus, to obtain a solution to the Gowdy model, we need to solve (2.8) and the evolution equations (2.4) and (2.5). However, as we will see, in the Hamiltonian framework, the global constraint generates just the rigid rotation around the circle. Therefore, it has rather simple Poisson brackets with other constraints and constants of motion. Hence, the heart of the problem lies in the evolution equations. We shall initially focus on these, and incorporate the global constraint at the end.

The equations, (2.4–2.5), may be derived from the action

$$S_G(W,\Phi) = \frac{1}{2}\int dtd\theta\,t\,[\,\dot{W}^2 - W'^2 + \sinh^2 W(\dot{\Phi}^2 - \Phi'^2)\,]$$

$$= \frac{1}{2}\int dtd\theta\,t\,\sqrt{-\eta}\,\eta^{ab}G_{AB}(Y)\,\partial_a Y^A \partial_b Y^B \, , \qquad (2.9)$$

where $Y^1 = W$, $Y^2 = \Phi$, $\eta^{ab} = \mathrm{diag}(-,+)$, $a, b, \cdots = t, \theta$, and $G_{AB}(Y)\,dY^A dY^B = dW^2 + \sinh^2 W\,d\Phi^2$ is the unit hyperboloid metric. The Gowdy model with one polarization is obtained by setting $\Phi = 0$ in the above equations. This leads to a linear evolution equation for W, and hence exact solvability.

B. Cylindrical gravitational waves

The space-time metric for cylindrical gravitational waves can be put into the general form (2.1–2.2) above. The line element is

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[1] We thank the referee for drawing our attention to this point.

$$ds^2 = e^{2A} \left(-dt^2 + dr^2\right) + g_{ab} \, dx^a dx^b \; . \tag{2.10}$$

The spatial topology is now R^3, the orbits of the two Killing vector fields has topology $S^1 \times R$, and all the metric functions depend only on t and r. Now the relevant gauge fixing is $R(t, r) = r$. The vacuum Einstein equations are very similar to (2.4–2.5); all the t and r factors are interchanged with appropriate sign changes. The evolution and constraint equations are

$$\ddot{W} - \frac{1}{r} \, W' - W'' + \sinh W \, \cosh W \, (\Phi'^2 - \dot{\Phi}^2) = 0 \; , \tag{2.11}$$

$$\ddot{\Phi} - \frac{1}{r} \, \Phi' - \Phi'' + 2 \, \frac{\cosh W}{\sinh W} \, (\dot{\Phi}\dot{W} - \Phi'W') = 0 \; , \tag{2.12}$$

and

$$A' + \frac{1}{4r} - \frac{r}{4} \, [\, \dot{W}^2 + W'^2 + \sinh^2 W \, (\dot{\Phi}^2 + \Phi'^2) \,] = 0 \; , \tag{2.13}$$

$$\dot{A} - \frac{r}{2} \, (\dot{W}W' + \sinh^2 W \, \dot{\Phi}\Phi') = 0 \; , \tag{2.14}$$

where now $' \equiv \partial/\partial r$. Since the spatial topology is now R^1, there is no longer a global constraint. We can therefore focus simply on the evolution equations. These follow from the action

$$S_C(W, \Phi) = \frac{1}{2} \int dt dr \; r \, [\, \dot{W}^2 - W'^2 + \sinh^2 W (\dot{\Phi}^2 - \Phi'^2) \,] \; . \tag{2.15}$$

The only difference in the Lagrangian densities of (2.9) and (2.15) is the interchange of the t and r factors.

The awkward feature of these actions is the explicit factor of t or r in the respective integrands. Without these factors, the actions would be those of the usual two-dimensional non–linear σ–model, and standard methods for obtaining complete sets of conserved currents would be applicable. In the present case, these procedures have to be modified.

We will discuss two possible avenues. The first is to embed the two dimensional model (2.9) in a standard three dimensional, flat space non-linear σ-model, which is free of the t (or r) factor. (The two-dimensional model, with its t (or r) factor is recovered as a dimensional reduction of the standard three dimensional model.) In this approach, the solutions of the reduced Einstein theory constitute a specific (symmetry-reduced) sector of the solution space of the three dimensional σ-model. The second avenue is to remain in two dimensions and handle the t (or r) factor by imposing a t (or r) dependent constraint on a free theory of three scalar fields. We will explore these avenues in sections III and IV respectively.

III. RELATION TO THREE DIMENSIONAL σ AND CHIRAL MODELS

In Section III.A, we point out a relation between $SL(2, R)$ or $SO(3)$ σ-models and principal chiral models for these groups which exists in *any* space-time dimension but which appears not to have been noted in the literature. In Section III.B, we return to the reduced Einstein system and show that its space of solutions can be naturally embedded in to the

space of solutions of the $SL(2,R)$ σ–model in three space-time dimensions. Thus, the three models are closely related; the reduced Einstein theory is a sub-case of the other two models. In particular, any method of showing integrability or extracting conserved quantities in three dimensional σ or chiral models is directly applicable to the reduced Einstein system.

A. σ–model and flat connections

For concreteness let us consider the $SL(2,R)$ non-linear σ-model in n-dimensional Minkowski space-time; with obvious sign changes, our discussion goes through step by step in the $SO(3)$ case and for Euclidean space-time signature. The model has the action

$$S_n(X,\lambda) = -\frac{1}{2} \int d^n x \ \sqrt{-\eta} \ \left[\ \eta^{ab}\partial_a X^i \partial_b X^j g_{ij} + \lambda \ (g_{ij}X^i X^j + 1) \ \right] \ , \tag{3.1}$$

where η^{ab} is the flat Lorentzian metric of signature $(-,+,\cdots,+)$, g_{ij} is the Cartan metric for $SL(2,R)$ of signature $\mathrm{diag}(++-)$, and $\lambda(x)$ is a Lagrange multiplier. The equations of motion are:

$$\Box X^i \equiv \eta^{ab}\partial_a\partial_b X^i = \lambda X^i \tag{3.2}$$

$$g_{ij}X^i X^j + 1 = 0 \,. \tag{3.3}$$

To see the relation to the principal chiral model, define a Lie algebra valued 1–form by

$$A^i_a = -2f^i{}_{jk}X^j\partial_a X^k, \tag{3.4}$$

where $f^i{}_{jk}$ are the $SL(2,R)$ structure constants. Then it follows from the equation of motion (3.2) that

$$\partial^a A^i_a = 0 \,. \tag{3.5}$$

Furthermore, using $X^i\partial_a X_i = 0$, which follows from the constraint (3.3), it is easy to show that

$$F^i_{ab} = \partial_a A^i_b - \partial_b A^i_a + f^i{}_{jk}A^j_a A^k_b = 0 \,. \tag{3.6}$$

Thus, the σ–model equations imply (3.5) and (3.6), which are the equations of the principal chiral model. The last step makes a crucial use of the three dimensionality of our Lie groups (i.e., the specific form of the structure constants of $SL(2,R)$ (or $SO(3)$)). Hence, while this relation holds in *any* space-time dimension, it does *not* hold for general groups.

Let us now ask if this map from the the space of solutions of the non-linear sigma model to that of the principal chiral model is surjective. Can we derive (3.2–3.3) from the chiral model equations? First, $F_{ab} = 0$ implies that A_a has the form $A_a = g^{-1}\partial_a g$ for $g \,\epsilon\, SL(2,R)$. Now, a standard parametrization of g is

$$g = \begin{pmatrix} \alpha + X^2 & X^1 + X^3 \\ X^1 - X^3 & \alpha - X^2 \end{pmatrix} \,, \tag{3.7}$$

with the unit determinant condition $\alpha^2 - g_{ij}X^i X^j = 1$. This gives

$$A_a = g^{-1}\partial_a g = 2\left(\alpha\partial_a\mathcal{X} - \mathcal{X}\partial_a\alpha\right) - 2\left[\mathcal{X}, \partial_a\mathcal{X}\right], \tag{3.8}$$

where $\mathcal{X} = X^i\tau_i$ is an element of $sl(2,R)$ with generators τ_i. Secondly, $\partial^a A_a = 0$ leads to the equation

$$[\mathcal{X}, \Box\mathcal{X}] = \alpha\Box\mathcal{X} - \mathcal{X}\Box\alpha. \tag{3.9}$$

The unit determinant condition implies that the σ-model constraint (3.3) is satisfied if and only if $\alpha = 0$. In this case, the commutator $[\mathcal{X}, \Box\mathcal{X}]$ vanishes which in turn implies that $\Box X^i = \lambda X^i$ for some function $\lambda(x)$. Thus, the σ-model equations hold if and only if the $SL(2,R)$ matrices g are trace free. To summarize, the space of solutions to the σ-model is naturally embedded in the space of solutions to the principal chiral model. The image consists of flat connections A_a given by $A_a = g^{-1}\partial_a g$ where g is trace-free (i.e., where $\alpha = 0$ in the parametrization (3.7)). We will return to this result in the next subsection.

To conclude this discussion, we note that there is also a Hamiltonian version of the derivation of the $SL(2,R)$ principal chiral model from the corresponding σ–model, which is presented below for a more general time dependent constraint $X^i X_i + t = 0$. As we will see, this is the type of constraint relevant for the reduced Einstein equations if one wants to work with two dimensional models.

B. Reduced theory from three dimensions

Let us now specialize to three-dimensional space-times with topology $S^1 \times R^2$, equipped with a flat metric η_{ab}, given by

$$\eta_{ab}dx^a dx^b = -d\tau^2 + dx^2 + d\theta^2. \tag{3.10}$$

Consider the action

$$S_3'[Y] = -\frac{1}{2}\int_M d\tau dx d\theta \sqrt{-\eta}\, \eta^{ab}G_{AB}(Y)\, \partial_a Y^A \partial_b Y^B, \tag{3.11}$$

where, as before $A = 1, 2$ but now $Y^A = Y^A(\tau, x, \theta)$. As is well-known, this action yields the same equations of motion as $S_3(X, \lambda)$ of (3.1). Note, however, that, in contrast with the reduced Einstein action (2.9), there is no factor of τ in the integrand of (3.11). However, we will now show that (2.9) does result from (3.11) by a symmetry reduction. Thus, the reduced Einstein model is contained in the three dimensional, standard, non-linear σ-model.

To perform the required reduction to two dimensions, we proceed in two steps. First, let us make the change of coordinates

$$\tau = t\, \cosh y, \qquad x = t\, \sinh y \tag{3.12}$$

in the $\tau - x$ plane, which casts the metric in the form

$$\eta_{ab}\, dx^a dx^b = -dt^2 + t^2 dy^2 + d\theta^2. \tag{3.13}$$

Second, let us require that the Lie derivative of the field variables $Y^A(t, y, \theta)$ along the boost-Killing vector field $(\partial/\partial y)^a$ of η^{ab} vanish. (In the chosen coordinates this means $Y^A = Y^A(t, \theta)$.) Then, we have:

$$S_3'[Y] \longrightarrow S_2'[Y] = -\frac{1}{2} \int dy \int dt d\theta \; t \; \eta^{ab} G_{AB}(Y) \; \partial_a Y^A \partial_b Y^B = S_G[Y] \left(\int dy \right) \;, \quad (3.14)$$

which is the Einstein action (2.9) for Gowdy models, multiplied by the constant $\int dy$.

Thus, we arrive at the following conclusions. As we have just shown, the space of solutions $\mathcal{S}_G$ to the reduced Einstein equations (i.e. to the Gowdy model) is naturally embedded *in* the space $\mathcal{S}_\sigma$ of solutions to the σ-model resulting from the action $S_3'[Y]$ of (3.11), or $S_3[X, \lambda]$ of (3.1). Second, we saw in section III.A that the space $\mathcal{S}_\sigma$ is in turn embedded into the space of solutions $\mathcal{S}_{ch}$ to the principal chiral model. The image of the first map consists of fields $Y^A(t, \theta)$ which are y-independent, and satisfy the global constraint (2.8), while the image of the second map consists of $A_a = g^{-1} \partial_a g$ for which the g of (3.7) is trace-free. Therefore, any method for finding conserved quantities for the standard non-linear σ-model or the principal chiral model in three dimensions yields, via restriction, a method for finding conserved quantities for the reduced Einstein equations. However, in practice, this simplification is not directly useful since the standard techniques for finding conserved currents for σ and chiral models are tailored to two space-time dimensions. Nonetheless, the existence of embeddings is conceptually interesting and may have a more general application. For example, results on integrability, asymptotic forms of solutions, and quantization of the standard three dimensional σ and chiral models will also apply to Gowdy models and cylindrical waves by restriction. Indeed, this is essentially the procedure used to (find the complete set of constants of motion and) quantize one polarization cylindrical [7,16], toroidal [5] waves and Gowdy models [24].

IV. RELATION TO TWO DIMENSIONAL GENERALIZED σ AND CHIRAL MODELS

Let us now explore the relation between the reduced Einstein system and *two* dimensional models. In three space-time dimensions, we could restrict ourselves to the standard σ and chiral models. In two dimensions, on the other hand, we will have to allow certain generalizations.

More precisely, in Sec. IV.A, we will show that the reduced Einstein equations can be derived from a two dimensional generalized σ–model action, where, however, the constraint is time (or space) *dependent*. In Sec. IV.B, we will perform a Legendre transform and show that the Hamiltonian evolution equations for certain variables give a generalized chiral model. This latter result is a Hamiltonian version of that given in Sec. III.A above, suitably generalized to incorporate the space or time dependent σ–model constraint.

A. Time dependent σ–model

Let the space-time topology be $S^1 \times R$ and let η_{ab} be the obvious flat metric with signature $(-, +)$. Consider the following two-dimensional action for three free scalar fields X^i ($i = 1, 2, 3$):

$$S[X, \lambda] = -\frac{1}{2} \int dt d\theta \left[\sqrt{-\eta} \; \eta^{ab} g_{ij} \; \partial_a X^i \partial_b X^j - \lambda \left(g_{ij} X^i X^j + t \right) \right] \;, \quad (4.1)$$

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where $g_{ij} =$ diag$(+ + -)$. The variation of S with respect to λ gives the *time dependent* constraint

$$C_1 := g_{ij}X^iX^j + t \equiv \bar{X} \cdot \bar{X} + t = 0 .\tag{4.2}$$

This constraint is solved by setting

$$X^1 = \sqrt{t} \cos\Phi \sinh W, \quad X^2 = \sqrt{t} \sin\Phi \sinh W, \quad X^3 = \sqrt{t} \cosh W ,\tag{4.3}$$

where the fields $W(t, \theta)$ and $\Phi(t, \theta)$ are unrestricted. These conditions, when substituted in to the action (4.1), give the reduced action

$$S_R[W, \Phi] = \frac{1}{2} \int dtd\theta \; \left\{ \; t \, [\; \dot{W}^2 - W'^2 + \sinh^2 W(\dot{\Phi}^2 - \Phi'^2) \;] - \frac{1}{4t} \right\}.\tag{4.4}$$

The integrand here differs from that of S_G of (2.9) only by an additive field independent term, and therefore leads to the same evolution equations as (2.9). It is also straightforward to verify directly that, as is expected on general grounds, the equations of motion following from $S[X, \lambda]$, together with (4.3), lead to the reduced Einstein equations (2.4–2.5).

B. Hamiltonian formulation and generalized chiral model

We will now show that the Legendre transform of the action (4.1) leads to "chiral model–like" equations, where however the curvature is not flat. The phase space form of the action is:

$$S[X, \lambda, P] = \int dtd\theta \left\{ \; P_i\dot{X}^i - \frac{1}{2} \; \left[g^{ij}P_iP_j + g^{ij}X_i'X_j' - \lambda \, (g^{ij}X_iX_j + t) \; \right] \right\} .\tag{4.5}$$

where P_i are the momenta conjugate to X^i. Following now the Dirac prescription, we require that the constraint C_1 of (4.2) be preserved in time:

$$\frac{dC_1}{dt} = \{C_1, H\} + \frac{\partial C_1}{\partial t} = 2\bar{X} \cdot \bar{P} + 1 = 0 ,\tag{4.6}$$

where

$$\begin{aligned} H &= \frac{1}{2} \int_0^{2\pi} d\theta \; g^{ij}(P_iP_j + X_i'X_j') + \frac{1}{2} \int_0^{2\pi} d\theta \; \lambda C_1 \\ &\equiv H_0 + \frac{1}{2} \int_0^{2\pi} d\theta \; \lambda C_1 \end{aligned}\tag{4.7}$$

is the Hamiltonian identified from (4.5). This gives the secondary constraint

$$C_2 := 2\bar{X} \cdot \bar{P} + 1 = 0.\tag{4.8}$$

The pair of constraints $C_1 = 0$ and $C_2 = 0$ are of second class. The constraint C_2 is solved by setting

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$$P_1 = \frac{1}{2\sqrt{t}} \cos\Phi \, \sinh W - \sqrt{t} \, \sin\Phi \, \sinh W \, \dot{\Phi} + \sqrt{t} \, \cos\Phi \, \cosh W \, \dot{W}$$

$$P_2 = \frac{1}{2\sqrt{t}} \sin\Phi \, \sinh W + \sqrt{t} \, \cos\Phi \, \sinh W \, \dot{\Phi} + \sqrt{t} \, \sin\Phi \, \cosh W \, \dot{W}$$

$$P_3 = -\frac{1}{2\sqrt{t}} \cosh W - \sqrt{t} \, \sinh W \, \dot{W} \tag{4.9}$$

and using (4.3). As we did for C_1, let us require that the secondary constraint C_2 be preserved in time. This will lead either to a new constraint, or to a condition on the lagrange multiplier λ. It turns out that the latter is the case :

$$\frac{dC_2}{dt} = \{C_2, H\} + \frac{\partial C_2}{\partial t} = \{C_2, H_0\} + 4t\lambda = 0 , \tag{4.10}$$

which gives $\lambda = -\{C_2, H_0\}/4t$ and

$$H = H_0 - \frac{1}{8t} \int_0^{2\pi} d\theta \, \{C_2, H_0\} C_1 . \tag{4.11}$$

This is the (first class) Hamiltonian we must use to derive evolution equations. The above procedure is equivalent to using the Dirac brackets for the unmodified Hamiltonian H_0. To summarize, a Hamiltonian description of the evolution equations of the Gowdy model can be given as follows. The phase space consists of canonical pairs (X^i, P_i). There are two second class constraints and the Hamiltonian is given by (4.11). If we initially satisfy the two constraints, they continue to be satisfied by the Hamiltonian flow.

Recall however that the equations governing the Gowdy model also includes a global constraint (2.8). It is straightforward to verify that this constraint generates the rigid diffeomorphism on S^1. Hence it weakly Poisson-commutes with C_1 and C_2 and strongly Poisson-commutes with the Hamiltonian. Therefore, it is a single (global) first class constraint within the Hamiltonian description arrived at above and we can just carry it along in the analysis. Constants of motion of the system *without* the global constraint will provide physical constants of motion simply by restriction to the part of the phase space where the global constraint is satisfied *provided* they are invariant under rigid diffeorphisms of S^1.

To go to a generalized chiral model, it is convenient to replace (X^i, P_i) pairs by new variables

$$L^i = f^i{}_{jk} X^j P^k \quad \text{and} \quad J^i = f^i{}_{jk} X^j X^{k\prime}, \tag{4.12}$$

where $f^i{}_{jk}$ are the $SL(2, R)$ structure constants. We will now argue that there is no loss of information in using L^i, J^i in place of X^i, P^i satisfying the two constraints (4.2) and (4.6). First, it is easy to verify that both L^i and J^i are space-like in the internal directions (but can vanish). Hence, $\bar{J} \times \bar{L}$ is time-like. When this last vector is non-zero, one can easily recover X^i and P^i from L^i and J^i explicitly:

$$\bar{X} = \pm\frac{\sqrt{t}}{\left|\bar{J} \times \bar{L}\right|} \, \bar{J} \times \bar{L} , \qquad \bar{P} = \pm\frac{1}{\sqrt{t}\left|\bar{J} \times \bar{L}\right|} \left(\frac{1}{2}\bar{J} \times \bar{L} + \left[(\bar{J} \times \bar{L}) \times \bar{L}\right]\right) , \tag{4.13}$$

where

$$\left| \bar{J} \times \bar{L} \right| = \sqrt{-(\bar{J} \times \bar{L}) \cdot \left(\bar{J} \times \bar{L} \right)} \ . \tag{4.14}$$

The choice of the sign in the two equations can be fixed by first requiring that X^i be future-directed and then using (4.6). Since $\bar{J} \times \bar{L}$ is time-like, its norm can vanish only when the vector itself vanishes. In this case, we can not simply use explicit formulas (4.13). However, a more detailed, systematic analysis shows that X^i, P^i can be again recovered from L^i, J^i.

L^i and J^i satisfy simple Poisson bracket relations

$$\begin{aligned}
&\{L_i(t, \theta), L_j(t, \theta')\} = f_{ij}{}^k L_k(t, \theta) \ \delta(\theta, \theta') \ , \\
&\{J_i(t, \theta), J_j(t, \theta')\} = 0 \ , \\
&\{L_i(t, \theta), J_j(t, \theta')\} = f_{ij}{}^k J_k(t, \theta) \ \delta(\theta, \theta') + g_{ij} \delta'(\theta, \theta') \ .
\end{aligned} \tag{4.15}$$

Furthermore,

$$\{L^i(t, \theta), C_1(t, \theta')\} = \{L^i(t, \theta), C_2(t, \theta')\} = 0 \tag{4.16}$$

$$\{J^i(t, \theta), C_1(t, \theta')\} = 0, \quad \{J^i(t, \theta), C_2(t, \theta')\} = 4J^i(t, \theta) \ \delta(\theta, \theta') \ . \tag{4.17}$$

(The indices $i, j, \cdots$ are lowered (raised) using the $SL(2, R)$ metric g_{ij}.) The evolution equations of the $SL(2, R)$ variables (L^i, J^i) are:

$$\dot{L}^i = \{L^i, H\} = J^{i\prime}, \tag{4.18}$$

where, as before, the prime denotes the derivative with respect to θ, and

$$\begin{aligned}
\dot{J}^i &= \{J^i, H_0\} - \{J^i, C_1\} \left(\frac{1}{4t} \right) \{C_2, H\} = \{J^i, H_0\} \\
&= L^{i\prime} - 2f^i{}_{jk} X^{j\prime} P^k \ .
\end{aligned} \tag{4.19}$$

The last equation can be further simplified by noting that

$$P^i \approx \frac{1}{2t} \ X^i + \frac{1}{t} \ f^i{}_{jk} X^j L^k, \tag{4.20}$$

$$\begin{aligned}
f^i{}_{jk} X^{j\prime} P^k &\approx -\frac{J^i}{2t} + \frac{1}{t} \ X^i(\bar{X}' \cdot \bar{L}) \\
&\approx -\frac{J^i}{2t} + \frac{1}{t} \ f^i{}_{jk} L^j J^k,
\end{aligned} \tag{4.21}$$

where $\approx$ denotes equality modulo the constraints C_1 and C_2. This finally gives

$$\dot{J}^i = L^{i\prime} - \frac{2}{t} f^i{}_{jk} L^j J^k + \frac{1}{t} J^i. \tag{4.22}$$

We are now ready to put these Hamiltonian equations in to a chiral model–like form. Define the matrices

$$A_0 := 2L^i \tau_i \ , \qquad A_1 := 2J^i \tau_i \ , \tag{4.23}$$

where

$$\tau_1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tau_2 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau_3 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{4.24}$$

are the generators of $SL(2,R)$ satisfying the relations

$$[\tau_i, \tau_j] = f_{ij}{}^k \tau_k, \qquad \tau_i \tau_j = \frac{1}{2} f_{ij}{}^k \tau_k. \tag{4.25}$$

Multiplying the evolution equations (4.18) and (4.22) by τ^i leads to

$$\partial_0 A_0 - \partial_1 A_1 = 0 \tag{4.26}$$

$$\partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] = \left(1 - \frac{1}{t}\right) [A_0, A_1] + \frac{A_1}{t}, \tag{4.27}$$

where $0, 1 = t, \theta$; $t > 0$ and $0 \le \theta < 2\pi$. This is the desired chiral model-like form of the Einstein evolution equations (2.4–2.5).

So far, we have considered the Gowdy model equations where the relevant σ–model constraint is time dependent. The analysis of cylindrical waves is completely analogous. The evolution equations (2.11–2.12) may be derived from an action like (4.1), but with r replacing t in the constraint. After performing a Hamiltonian decomposition, and following steps similar to the above, we find that the cylindrical wave evolution equations for the variables L_i and J_i lead to the chiral model–like equations

$$\partial_0 A_0 - \partial_1 A_1 = 0, \tag{4.28}$$

$$\partial_0 A_1 - \partial_1 A_0 + [A_0, A_1] = \left(1 - \frac{1}{r}\right) [A_0, A_1] + \frac{A_0}{r}. \tag{4.29}$$

Note that these are different from the Gowdy model equations (4.26–4.27) only in the replacement $1/t$, $A_1/t \to 1/r$, A_0/r. In addition, now, there is no analog of the global constraint (2.8) to worry about.

V. CONSTANTS OF MOTION

For the chiral model in two space-time dimensions, there exist standard techniques to construct an infinite hierarchy of conserved currents. In this section, we will show that one of them admits a suitable extension which is applicable to the generalized chiral model of Sec. IV.

A. Gowdy cosmology

The idea is to use the Hamiltonian equations (4.26–4.27) to extract constants of motion for the reduced Einstein equations. The method is a modification of one given by Brezin et. al. (BIZZ) for obtaining constants of motion of the principal chiral model [25]. Let us first rewrite the evolution equation (4.27) in a convenient form:

$$F_{01} = A_1 + (1 - t)\epsilon^{ab} \partial_a A_b. \tag{5.1}$$

In the standard chiral model, the right side vanishes. The divergence-free condition (4.26) on A_a is the same in the two cases. So, modification of the (BIZZ) procedure of [25] is necessary to handle the right side of (5.1).

As for the principal chiral model, (4.26) is already a conservation equation, so we can write the first conserved current as

$$J_a^{(1)} = D_a \lambda^{(0)} \, , \tag{5.2}$$

where $D_a = \partial_a + A_a$ and $\lambda^{(0)}$ is any constant matrix. Denote the corresponding conserved charge by $Q^{(1)}$:

$$Q^{(1)} = \int_0^{2\pi} d\theta \ J_0^{(1)} = \int_0^{2\pi} d\theta \ \left[A_0, \lambda^{(0)} \right] \tag{5.3}$$

Since $J_a^{(1)} - (Q^{(1)}/2\pi)\partial_a t =: J_a^{(1)} - \partial_a \phi^{(1)}$ is a smooth conserved current with zero charge, it is dual of the gradient of a smooth field. Thus, there exists a smooth, matrix-valued field $\lambda^{(1)}$ such that

$$J_a^{(1)} - \partial_a \phi^{(1)} = \epsilon_a{}^b \partial_b \lambda^{(1)} \, . \tag{5.4}$$

This equation determines $\lambda^{(1)}$ up to an additive constant matrix:

$$\lambda^{(1)}(\theta, t) = -\int_0^\theta d\theta' \ J_0^{(1)}(\theta', t) + \frac{Q^{(1)}\theta}{2\pi} - \int_{t_0}^t dt' \ J_1^{(1)}(0, t') \, . \tag{5.5}$$

Note that, in spite of the explicit appearance of θ in the second term, the 'potential' $\lambda^{(1)}$ is smooth everywhere, including the point $\theta = 0$. It depends on A_a, i.e., on the solution to the Gowdy model under consideration and the idea is to construct another conserved current from $\lambda^{(1)}$.

As in the (BIZZ) procedure of [25], let us introduce a fiducial current,

$$K_a^{(2)} := D_a \lambda^{(1)} \, , \tag{5.6}$$

and compute its divergence

$$\begin{aligned}
\eta^{ab}\partial_a K_b^{(2)} = \eta^{ab} D_b \partial_a \lambda^{(1)} &= \epsilon^{ab} D_a \left(J_b^{(1)} - \partial_b \phi^{(1)} \right) \\
&= \epsilon^{ab} D_a D_b \lambda^{(0)} - \epsilon^{ab} \left[A_a, \partial_b \phi^{(1)} \right] \\
&= \left[F_{01}, \lambda^{(0)} \right] - \epsilon^{ab} \left[A_a, \partial_b \phi^{(1)} \right] \\
&= \left[A_1, \lambda^{(0)} \right] + (1-t)\epsilon^{ab} \left[\partial_a A_b, \lambda^{(0)} \right] - \epsilon^{ab} \left[A_a, \partial_b \phi^{(1)} \right] \\
&= \epsilon^{ab}\partial_a \left[(1-t)A_b, \lambda^{(0)} \right] + \epsilon^{ab}\delta_a^0 \left[A_b, \lambda^{(0)} \right] + \left[A_1, \lambda^{(0)} \right] - \epsilon^{ab} \left[A_a, \partial_b \phi^{(1)} \right] \, , \quad (5.7)
\end{aligned}$$

where the equation of motion (5.1) is used in the fourth equality. Collecting terms, we obtain

$$\eta^{ab}\partial_a K_b^{(2)} = \epsilon^{ab}\partial_a \left[(1-t)A_b, \lambda^{(0)} \right] + 2\epsilon^{ab}\delta_a^0 \left[A_b, \lambda^{(0)} \right] + \epsilon^{ab}\delta_a^0 \left[A_b, \frac{Q^{(1)}}{2\pi} \right] . \tag{5.8}$$

Now the key observation is that the last two terms on the r.h.s. may be rewritten as divergences. Indeed, using (5.4), we obtain

$$\epsilon^{ab}\delta_a^0 \left[A_b, \lambda^{(0)}\right] = \epsilon^{ab}\delta_a^0 \left(\epsilon_b{}^c\partial_c\lambda^{(1)} + \partial_b\phi^{(1)}\right) = \eta^{ab}\partial_a \left(\lambda^{(1)}\delta_b^0 + \phi^{(1)}\delta_b^1\right) . \tag{5.9}$$

Similarly, $[A_a, Q^{(1)}/2\pi]$ is a conserved current. Using it in place of $J_a^{(1)} = [A_a, \lambda^{(0)}]$ in the steps that led us to Eqn. (5.5), we obtain

$$\epsilon^{ab}\delta_a^0 \left[A_b, \frac{Q^{(1)}}{2\pi}\right] = \epsilon^{ab}\delta_a^0 \left(\epsilon_b{}^c\partial_c\bar{\lambda}^{(1)} + \partial_b\bar{\phi}^{(1)}\right) = \eta^{ab}\partial_a \left(\bar{\lambda}^{(1)}\delta_b^0 + \bar{\phi}^{(1)}\delta_b^1\right) . \tag{5.10}$$

Here $\bar{\lambda}^{(1)}$ is defined similarly to $\lambda^{(1)}$ of Eq. (5.5) by

$$\bar{\lambda}^{(1)}(\theta, t) = -\int_0^\theta d\theta' \left[A_0(\theta', t), Q^{(1)}/2\pi\right] + \frac{\bar{Q}^{(1)}\theta}{2\pi} - \int_{t_0}^t dt' \left[A_1(0, t'), \frac{Q^{(1)}}{2\pi}\right] , \tag{5.11}$$

with $\left[A_a, Q^{(1)}/2\pi\right]$ replacing $J_a^{(1)} = \left[A_a, \lambda^{(0)}\right]$, and $\bar{\phi}^{(1)} = \bar{Q}^{(1)}t/2\pi$ with

$$\bar{Q}^{(1)} = \int_0^{2\pi} d\theta \left[A_0(\theta, t), Q^{(1)}/2\pi\right] . \tag{5.12}$$

Thus, we have shown that the current

$$J_a^{(2)} \equiv K_a^{(2)} - \epsilon_a{}^b(1-t)\left[A_b, \lambda^{(0)}\right] - \delta_a^0\left(2\lambda^{(1)} + \bar{\lambda}^{(1)}\right) - \delta_a^1\left(2\phi^{(1)} + \bar{\phi}^{(1)}\right) \tag{5.13}$$

is conserved. The corresponding conserved charge is

$$Q^{(2)} = \int_0^{2\pi} d\theta\, J_0^{(2)}(\theta, t) = \int_0^{2\pi} d\theta \left\{ D_0\lambda^{(1)} - \left(2\lambda^{(1)} + \bar{\lambda}^{(1)}\right) + (1-t)\left[A_1, \lambda^{(0)}\right] \right\} , \tag{5.14}$$

with $\lambda^{(1)}$ and $\bar{\lambda}^{(1)}$ are defined as in (5.5) and (5.11).

Notice that $Q^{(1)} = Q_i^{(1)}\tau_i$ and $Q^{(2)} = Q_i^{(2)}\tau_i$ are independent charges because the latter depends explicitly on t, A_0 and A_1, whereas the former depends only on A_0. These are in fact six different charges because we can "peel off" the matrices τ_i. By inspection, these charges are invariant under rigid diffeomorphisms of S^1. Hence, they Poisson-commute with the global constraint (2.8) and are physical constants of motion for the Gowdy model.

The key question now is whether we can build a whole tower of conserved charges. A natural strategy is to use Poisson brackets between the $Q_i^{(1)}$ and $Q_i^{(2)}$. The brackets between $Q_i^{(1)}$ among themselves form the $sl(2, R)$ Lie algebra; they do not yield new charges. Moreover, their brackets with $Q^{(2)}$ just rotate the $Q_i^{(2)}$ among themselves. On the other hand, the Poisson brackets $\{Q_i^{(2)}, Q_j^{(2)}\}$ provides charges whose expressions have a higher degree of non-locality than the $Q_i^{(2)}$ themselves; the expressions contain one more nested integral. It is therefore reasonable to suppose that, generically, the Poisson bracket is a *new* conserved charge. It is straightforward to see that the continuation of this process gives conserved charges of successively higher degrees of non-locality, each of which appear to be functionally independent of the previous ones. Thus, the procedure yields an infinite tower of conserved quantities. Although we do not have a conclusive proof, the structure of non-locality suggests that all these charges are independent. Finally, by construction, all these charges Poisson-commute with the global constraint (2.8) and are therefore physical charges for the Gowdy model.

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Let us summarize. The equations of motion following from the action (4.1) are the reduced Einstein equations. We performed a $1+1$–decomposition of this action to obtain the Hamiltonian theory, and then made a change of variables from (X^i, P_i) to A_a^i. The equations for A_a^i resemble those of the chiral model. Hence, it is possible to extend the standard procedure [25] to obtain two sets of conserved charges $Q_i^{(1)}$ and $Q_i^{(2)}$. A tower of new conserved quantities can be obtained by taking Poisson brackets. Finally, note that the constants of motion can be rewritten in terms of the original metric variables (W, Φ) and their space and time derivatives by using the solutions (4.3) and (4.9) of the constraints C_1 and C_2.

B. Cylindrical waves

As noted before, the evolution equations for cylindrical waves and the T^3 Gowdy model are very similar. Therefore it is not surprising that the above approach to finding constants of motion applies to cylindrical waves. There are two main differences. First, the spatial topology is now R^3 rather than T^3, whence the spatial sections of the reduced two-dimensional model are half-lines $0 \le r$ rather than circles. The triviality of this topology will simplify the analysis. The second difference is that we now need to impose boundary conditions on our dynamical variables. For simplicity, we will assume that, on an arbitrarily chosen initial Cauchy surface, the fields W and Φ are of compact support. This assumption will ensure the convergence of various integrals but can be weakened in an obvious manner.

As in the Gowdy model, the existence of the first conserved charge follows immediately from (4.28). The conserved current has the same form as before: $J_a^{(1)} \equiv D_a \mu^{(0)}$, where $\mu^{(0)}$ is any constant, fiducial matrix. However, because the spatial topology of the reduced model is now trivial, the potential $\mu^{(1)}$ is now defined simply by

$$J_a^{(1)} = \epsilon_a{}^b \partial_b \mu^{(1)} \,, \tag{5.15}$$

which determines $\mu^{(1)}$ up to an additive constant matrix,

$$\mu^{(1)} = -\int_0^r dr' \, J_0^{(1)}(r', t) - \int_0^t dt' \, J_1^{(1)}(0, t') \,. \tag{5.16}$$

The second, fiducial current has the same form as $J_a^{(1)}$

$$\kappa_a^{(2)} := D_a \mu^{(1)} \,. \tag{5.17}$$

Taking its divergence, we find

$$
\begin{aligned}
\eta^{ab} \partial_a \kappa_b^{(2)} &= \epsilon^{ab} D_a J_b^{(1)} = \epsilon^{ab} D_a D_b \mu^{(0)} \\
&= \left[F_{01}, \mu^{(0)} \right] = (1-r)\epsilon^{ab} \left[\partial_a A_b, \mu^{(0)} \right] + \left[A_0, \mu^{(0)} \right] \\
&= \epsilon^{ab} \partial_a \left[(1-r)A_b, \mu^{(0)} \right] \\
&= \eta^{ab} \partial_a \left[(1-r)A_c, \mu^{(0)} \right] \epsilon_b{}^c \,.
\end{aligned}
\tag{5.18}
$$

This allows us to identify the second conserved current

$$J_a^{(2)} = \kappa_a^{(2)} - (1 - r)\epsilon_a{}^b \left[A_b, \mu^{(0)} \right] \ . \tag{5.19}$$

Thus, the triviality of spatial topology simplifies the analysis. The corresponding conserved charge is

$$Q^{(2)} = \int_0^\infty dr \ J_0^{(2)} = \int_0^\infty dr \left\{ D_0 \mu^{(1)} + (1 - r) \left[A_1, \mu^{(0)} \right] \right\} \ . \tag{5.20}$$

As before, further conserved quantities can be generated by taking Poisson brackets. Finally, in this case there is no global constraint to take care of.

VI. DISCUSSION

Let us summarize the main results. In Sec. III, we first showed that the standard, $SL(2, R)$ non-linear σ-model is embedded in the principal chiral model in any space-time dimension. We then showed that the symmetry reduced Einstein system is embedded in the σ model in *three* space-time dimensions. Thus, there is a hierarchy and results from the three-dimensional σ and chiral models can be taken over to the reduced Einstein system. This strategy is successful in the one-polarization case, i.e., the case when the two Killing fields are hypersurface-orthogonal.

In Sec IV we worked in two space-time dimensions and recast the reduced Einstein model as a 'time dependent' σ–model, or, alternatively, 'generalized' chiral model. For the standard chiral model, there exist the so-called BIZZ procedure which enables one to construct a hierarchy of conserved charges. In Sec. V, we showed that the procedure can be appropriately modified to the generalized chiral model of Sec. V to obtain the analogs of the first two BIZZ conserved currents. Modifications were required because of two reasons: i) the two-dimensional space-time topology is $S^1 \times R$, rather than R^2; and ii) the curvature of the generalized model does not vanish, but has a specific form. (To our knowledge, the complications that arise due to non-trivial topology have not been discussed in the literature, even in the standard BIZZ procedure.) Since the currents take values in the Lie-algebra of $SL(2, R)$, we obtain six conserved quantities, $Q_i^{(1)}$ and $Q_i^{(2)}$, where the index i refers to the Lie-algebra. One can restrict this procedure to the well-understood, one-polarization case. Even in this case, the second, non-local set of charges $Q_i^{(2)}$ appears to be new, i.e., appears to be a non-trivial combination of the known charges.

Returning to the general case, Poisson brackets between the $Q_i^{(2)}$ lead to new conserved charges and one can continue the process by taking Poisson brackets between the available charges. The functional form of these charges exhibit increasing non-locality, each step leading to an additional 'nested integral'. The initial-value problem for these equations is well-posed, and each charge appears to probe a different aspect of the functional form of the initial data. Therefore, it appears that the charges are all functionally independent. However, we do not have a definitive proof of this independence. For example, after a certain stage, the Poisson bracket may just yield a c-number, i.e., a constant functional on the space of solutions. In this case, the procedure would terminate. It may also happen that at a certain stage the conserved charges cease to be functionally differentiable, making it impossible to compute further Poisson brackets. However, the non-local structure of charges is reminiscent of the structure of the commutators in the Geroch group. Is there a close

relation between the two? If so, one would have an elegant avenue to show that the charges are indeed independent.

Finally, we should emphasize that, as matters stand, our results have not lead to a comprehensive treatment of issues like exact integrability or quantization in the full, two-polarization case. Rather, they provide new windows to tackle these issues. In particular, the techniques introduced in Secs. III and V should enable one to analyze the reduced Einstein system along lines that are quite different from the traditional ones.

Note added: After this work was completed and posted on the LANL archives (gr/qc 9712053), we were informed of a paper by Romano and Torre [26] which contains a result which is equivalent to that presented in Sec. III.B. Their main interest is in the 'issue of time' and they work in a Hamiltonian framework with a phase space which is enlarged to incorporate a 'clock degree of freedom'. As a side-remark, they point out that their Hamiltonian description can be derived from a symmetry reduced three-dimensional harmonic map in the parametrized field theory formalism.

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